

Structure of the Abel-Jacobi morphism and geometric local class field theory

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Lecture 4

$\mathrm{Spa}(E)^\diamond$

Sheaf of unitalts over E

\hookrightarrow on $\mathrm{Perf}_{\mathbb{F}_q}$

$$S(\mathrm{Perf}_{\mathbb{F}_q}), \quad \mathrm{Spa}(E)^\diamond(S) = \left\{ (S^\#, \iota) \mid S^\#(\mathrm{Perf}_E, \iota: S \xrightarrow{\sim} S^{\#,\flat}) \right\} / \sim$$

\rightarrow quickly said: " $S^\#$ is an unitalts of S "

$\mathrm{Spa}(E)^\diamond$ is a pro-étale sheaf via Scholze's equivalence

$$(-)^\flat: \mathrm{Perf}_{S^\#} \xrightarrow{\sim} \mathrm{Perf}_S$$

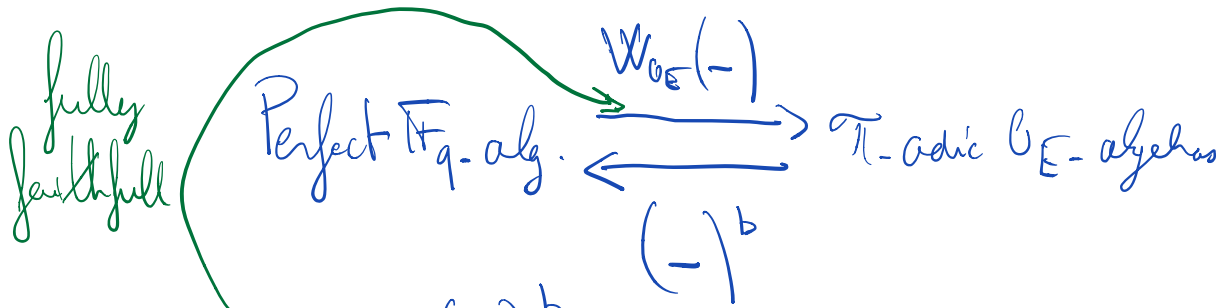
of pro-étale sites. Moreover it induces

$$\widetilde{\mathrm{Perf}}_E \xrightarrow{\sim} \widetilde{\mathrm{Perf}}_{\mathbb{F}_q} / \mathrm{Spa}(E)^\diamond$$

\mathcal{L}_n : $E \cong \mathbb{Q}_p$. $\widetilde{\text{Perf}}_{\mathbb{Q}_p} \cong \widetilde{\text{Perf}}_{\mathbb{F}_q} / \text{Spa}(\mathbb{Q}_p)^\diamond$ char. 0 pro. étale top.
 localization of char. p pro étale top.

Construction of a morphism $\text{Spa}(E)^\diamond \rightarrow \text{Div}^1$

An adjunction: E/\mathbb{Q}_p



$$\left\{ \begin{array}{l} \text{Id} \xrightarrow{\sim} W_{G_E}(-)^b \\ x \mapsto ([x^{h^m}])_{m \geq 0} \end{array} \right.$$

Fonkaine's \mathcal{D} -morphism
 = adjunction morphism

$$\left\{ \begin{array}{l} W_{G_E}(-)^b \xrightarrow{\mathcal{D}} \text{Id} \end{array} \right.$$

$$\left\{ \begin{array}{l} \sum_{m \geq 0} [x_m] \pi^m \mapsto \sum_{m \geq 0} x_m^\# \pi^m \end{array} \right.$$

* R perfectoid \mathbb{F}_q -algebra

$\xi = \sum_{m \geq 0} [x_m] \pi^m \in W_{G_E}(R^\circ)$ is said to be

primitive of degree 1 if $\begin{cases} x_0 \in R^{\circ\circ} \cap R^{\times} \\ x_1 \in (R^{\circ})^{\times} \end{cases}$

Prop. (1) $R^{\#}$ untilt of R over E , $R = R^{\#,b}$

$\mathcal{D}: W_{0,E}(R^{\circ}) \rightarrow R^{\#,0}$ is surjective
with ker $\mathcal{D} = (\xi)$, ξ primitive of degree 1.

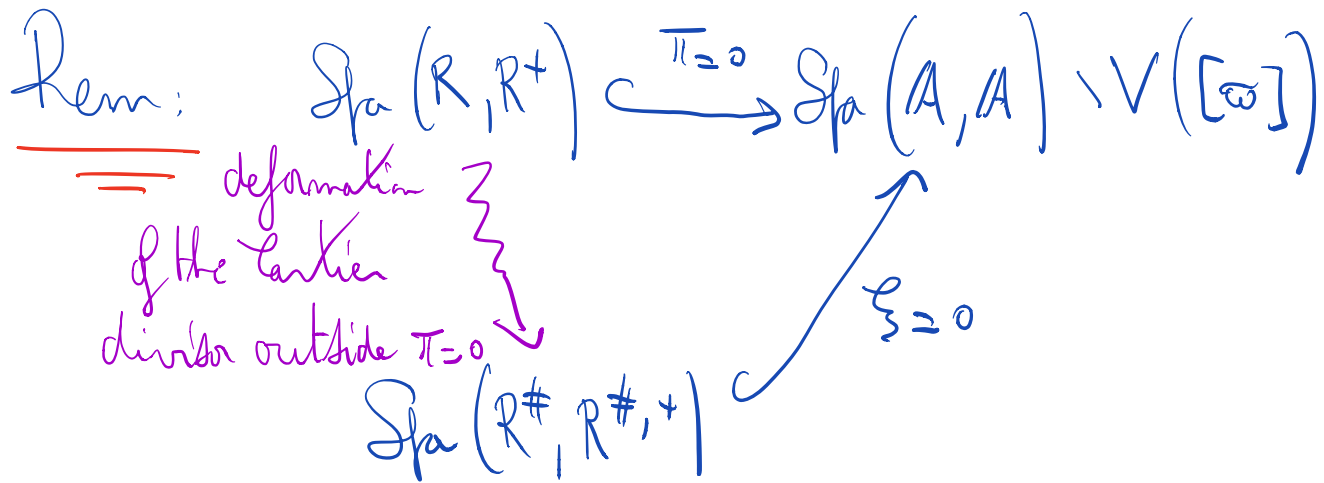
(2) Reciprocally, if $\xi \in W_{0,E}(R^{\circ})$ is primitive of degree 1 then $W_{0,E}(R^{\circ}) / (\xi) \left[\frac{1}{\xi} \right]$ is an untilt of R .

$$\Rightarrow \text{Spa}(E)^{\diamond}(R, R^+) = \left\{ \text{deg. 1 primitive} \right\} / W_{0,E}(R^{\circ})^{\times}$$

* $\xi \in \mathcal{O}(Y_{R,R^+})$ and $\times \xi: \mathcal{O}_{Y_{R,R^+}} \hookrightarrow \mathcal{O}_{Y_{R,R^+}}$

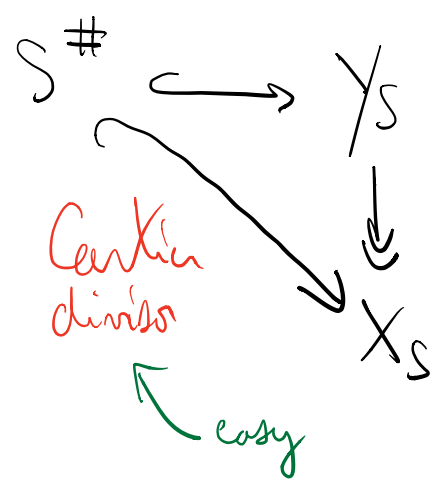
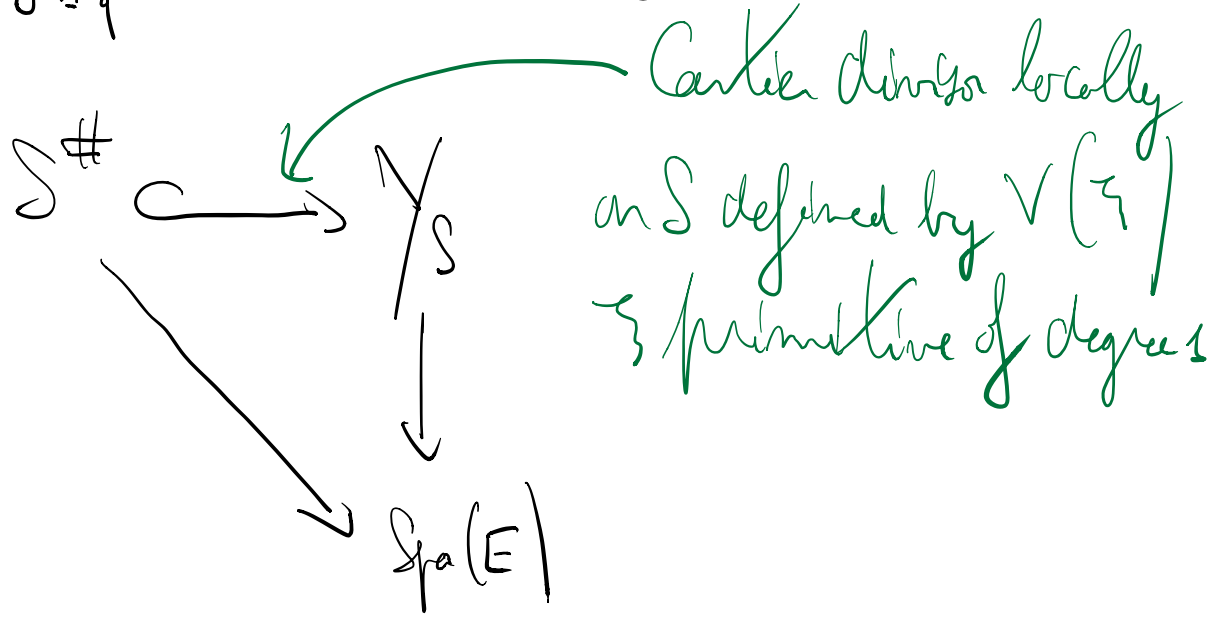
$$\text{Spa}(R^{\#}, R^{\#,+}) \xrightarrow{V(\xi)} Y_{R,R^+}$$

Cartier divisor

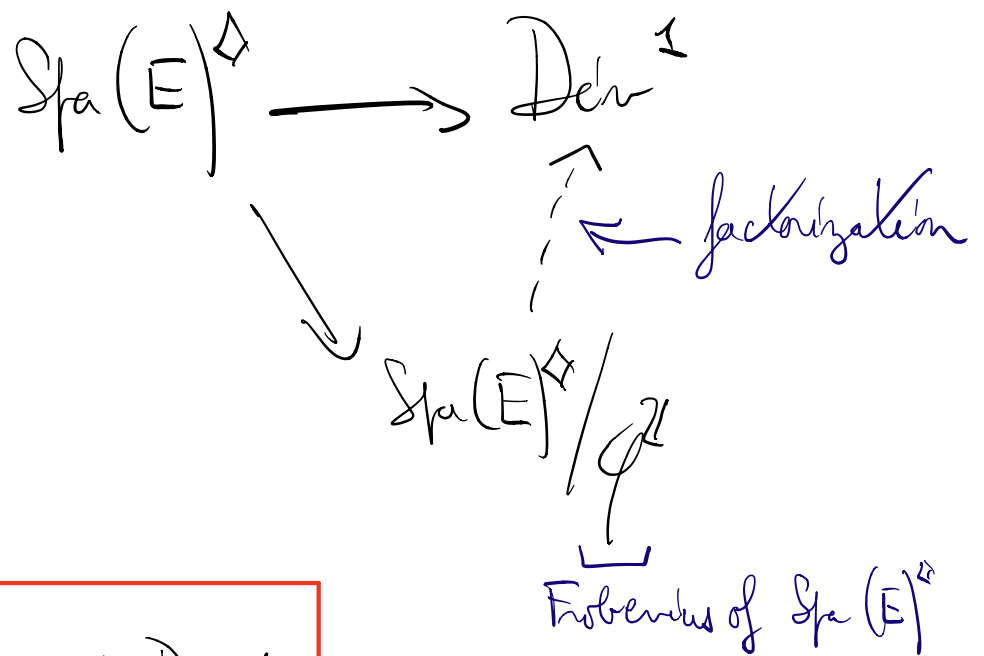


* $S \in \text{Perf}_{\mathbb{F}_q}$

$S^\#$ unlift of S over E



This defines a morphism



$$\text{Th: } \text{Spa}(E)^\diamond / \mathcal{O}^\times \xrightarrow{\sim} \text{Div}^1$$

Div^d:

$$\text{Th: } \Sigma^d : (\text{Div}^1)^d \longrightarrow \text{Div}^d \text{ induces}$$

$$\text{an isomorphism } (\text{Div}^1)^d / \mathcal{O}_d \xrightarrow{\sim} \text{Div}^d$$

quotient of pro-étale sheaves

→ proof technical: uses the notion of quasi-pro-étale morphism by Scholze. At the end this is reduced to prove that if F/\mathbb{F}_q alg. closed:

$$\text{Div}^1(\mathbb{F})^d / \sigma_d \cong \text{Div}^d(\mathbb{F})$$

deduced from the following factorization theorem, not Fontaine:

$$P = \bigoplus_{d \geq 0} \mathcal{O}(Y_{\mathbb{F}})^{\mathbb{F}_q = \pi^d} \text{ is graded factorial}$$

with deg. 1 irreducible elements

$$\rightsquigarrow \forall n \in P_d \setminus \{0\} \quad n = t_1 \dots t_d, \quad t_i \in P_1$$

(t_1, \dots, t_d) well defined up to permutation and multiplication by an element of E^\times .

$$\Rightarrow \forall d \geq 1, \text{Div}^d = \text{diamond (alg. space for pro. (kale top.)}$$

Qe - local systems on $\text{Div}_{\mathbb{F}_q}^1$:

$$\begin{aligned} \mathcal{E} &= \widehat{E^{\text{un}}} \\ \mathcal{G} & \\ \sigma &= \text{Fib} \end{aligned}$$

$$\text{Spa}(E)^{\diamond} \times_{\text{Spa}(\mathbb{F}_q)} \text{Spa}(\overline{\mathbb{F}_q}) = \text{Spa}(\mathcal{E})^{\diamond}$$

$$\mathcal{G}_{E^{\diamond}} \times \text{Id} \text{ and } \text{Id} \times \mathcal{G}_{\overline{\mathbb{F}_q}} \leftrightarrow \sigma$$

Composite of both partial Fib of absolute

Frob acts trivially on the étale topos.

$$\begin{aligned} \Rightarrow \overline{\mathcal{O}_E}\text{-loc. sys.} / \text{Div}_{\overline{\mathbb{F}_q}}^1 &= (\mathcal{O}_{E^{\text{ét}}} \times \text{Id})\text{-eq. loc. sys.} \\ &\text{on } \text{Spa}(E)^{\text{ét}} \times \text{Spa}(\overline{\mathbb{F}_q}) \\ &= \mathcal{O}\text{-eq. loc. sys. on } \text{Spa}(\overline{E})^{\text{ét}} \\ &= \boxed{\text{Rep}_{\overline{\mathcal{O}_E}}(W_E)} \end{aligned}$$

The Abel-Jacobi map

$$d \geq 1 \quad \text{AJ}^d: \text{Div}^d \rightarrow \text{Pic}^d$$

Another formula for Div^d :

$$\begin{aligned} \mathcal{B} &= \text{pro-étale sheaf of rings on } \text{Perf}_{\mathbb{F}_q} \\ \mathcal{B}(S) &= \mathcal{O}(Y_S) \\ &\text{sheaf of Fontaine's type of ring} \end{aligned}$$

$\mathbb{B}^{\mathcal{F}=\pi^d}$ = sheaf of relative global sections of $\mathcal{O}(d)$

$$\mathbb{B}(S)^{\mathcal{F}=\pi^d} = H^0(X_S, \mathcal{O}(d))$$

Pro. étale locally any \mathcal{L}/X_S of deg. d is isomorphic to $\mathcal{O}(d)$

$$\Rightarrow \text{Div}^d = \underbrace{\left(\mathbb{B}^{\mathcal{F}=\pi^d} \setminus \{0\} \right)}_{\text{Aut}(\mathcal{O}(d))} / \underline{\mathbb{E}^x}$$

Thus

$$AJ^d : \underbrace{\mathbb{B}^{\mathcal{F}=\pi^d} \setminus \{0\} / \underline{\mathbb{E}^x}}_{\text{Div}^d} \longrightarrow \underbrace{\left[\text{Spa}(\mathbb{F}_q) / \underline{\mathbb{E}^x} \right]}_{\text{Pic}^d}$$

One deduces

Th: $\forall d \geq 1$ AJ^d is a pro-étale locally
 $\underline{=}$ trivial fibration in $\underbrace{\mathbb{B}^{\mathcal{F}=\pi^d} \setminus \{0\}}_{\text{diamond}}$

Geometric Langlands

Let us admit (difficult)

Th: $\forall d \geq 2$ (resp. $d \geq 3$ if $E|_{\mathbb{Q}_r}$) $B_{\overline{\mathbb{F}_q}} \setminus \{0\}$
is simply connected (any finite étale cover has a section).

Let $\chi: W_E \rightarrow \overline{\mathbb{Q}_e}^\times$

$\rightsquigarrow \mathcal{E} = \text{nb. 1 } \overline{\mathbb{Q}_e}\text{-local system on } \text{Div}_{\overline{\mathbb{F}_q}}^1$

$\rightsquigarrow \mathcal{E}^{(d)} = \left(\sum_*^d \mathcal{E}^{\boxtimes d} \right)^{\otimes d}$ nb. 1 $\overline{\mathbb{Q}_e}$ -loc. sys
on $\text{Div}_{\overline{\mathbb{F}_q}}^d$

$\rightsquigarrow \mathcal{F}_r^{(d)}$ on Pic^d for $d \gg 0$, $\mathcal{E}^{(d)} = (A^d)^{\otimes \mathcal{F}_r^{(d)}}$

\rightsquigarrow group law on $\text{Pic} \Rightarrow \mathcal{F}_r^{(d)}$ extends

to all d 's to a local system on Pic .

In particular \mathcal{E} descends along AJ^1

What does it mean?

$$AJ^1: \text{Div}_{\overline{\mathbb{F}}_q}^1 \longrightarrow [\text{Spa}(\overline{\mathbb{F}}_q) / \underline{E}^x]$$

$$\pi_1(AJ^1): W_E \xrightarrow{f} E^x$$

no good theory of π_1 a priori but makes sense locally at the level of pullbacks of $\overline{\mathcal{O}_E}$ -local systems.

$$\chi_{\mathcal{L}_T}: W_E \rightarrow G_E^x \quad \text{Lubin-Tate character associated to } \pi$$

$$f(\tau) = \chi_{\mathcal{L}_T}(\tau)^{-1} \pi^{\nu(\tau)}$$

$$\tau \bmod \mathfrak{m} = \text{Frob}_q^{\nu(\tau)}$$

thus $\forall \chi: W_E \rightarrow \overline{\mathcal{O}_E}^x$, χ factorizes through

$$\Rightarrow \boxed{f: W_E^{\text{ad}} \xrightarrow{\sim} E^x} \quad (f \text{ is easily surjective})$$

i.e. local class field theory.

Proof that $B_{\overline{\mathbb{F}_q}}^{\varphi=\pi^d} \setminus \{0\}$ is simply connected when $d > 1$ for $\overline{\mathbb{F}_q}(\!(t)\!)$

$(\mathbb{R}, \mathbb{R}^+)$ $\overline{\mathbb{F}_q}$ -alg. aff. perfectoid

$\|\cdot\|: \mathbb{R} \rightarrow \mathbb{R}^+$ power mult. defining its topology

$$\mathcal{O}(Y_{\mathbb{R}, \mathbb{R}^+}) = \left\{ \sum_{m \in \mathbb{Z}} \chi_m \pi^m / \chi_m \in \mathbb{R} \forall \rho \in]0, 1[\quad \lim_{|m| \rightarrow \infty} \|\chi_m\| \rho^m = 0 \right\}$$

$$(\mathbb{R}^{\circ\circ})^d \xrightarrow{\sim} \mathcal{O}(Y_{\mathbb{R}, \mathbb{R}^+})^{\varphi=\pi^d}$$

$$(\chi_0, \dots, \chi_{d-1}) \mapsto \sum_{i=0}^{d-1} \sum_{b \in \mathbb{Z}} \chi_i^{q^{-b}} \pi^{bd+i}$$

$$\Rightarrow B_{\overline{\mathbb{F}_q}}^{\varphi=\pi^d} \simeq \text{Spa} \left(\overline{\mathbb{F}_q} \left[\chi_0^{1/q^\infty}, \dots, \chi_{d-1}^{1/q^\infty} \right] \right)$$

perfect adic space not perfectoid since not analytic

$$\mathbb{B}_{\overline{\mathbb{F}_q}}^{\varphi = \pi^d} \setminus \{0\} = \text{Spa}(\overline{\mathbb{F}_q}[[x_0^{1/\pi^\infty}, \dots, x_{d-1}^{1/\pi^\infty}]]) \setminus V(x_0, \dots, x_{d-1})$$

= perfectoid quasi-compact space

$$= \bigcup_{i=0}^{d-1} D(x_i)$$

↓ open perfectoid ball

$$\text{Spa}(\overline{\mathbb{F}_q}((x_i^{1/\pi^\infty})))$$

↪ Union of d open perfectoid balls / different perfectoid fields

GAGA: Th. Annoetherian \mathbb{F} -adic

$$\text{Finite étale} / \text{Spec}(A) \setminus V(\mathfrak{I}) \xrightarrow{\sim} \text{Finite étale} / \text{Spa}(A, A) \setminus V(\mathfrak{I})$$

$$\varinjlim_n \text{f. ét.} / \text{Spa}(\overline{\mathbb{F}_q}[[x_0^{1/\pi^n}, \dots, x_{d-1}^{1/\pi^n}]]) \setminus V(x_0, \dots, x_{d-1})$$

↓ \simeq Elbits deCompletion

$$\text{f. ét.} / \text{Spa}(\overline{\mathbb{F}_q}[[x_0^{1/\pi^\infty}, \dots, x_{d-1}^{1/\pi^\infty}]]) \setminus V(x_0, \dots, x_{d-1})$$

\leadsto suffices to prove that $\text{Spa}(\overline{\mathbb{F}_q}[[x_0, \dots, x_{d-1}]]) \setminus V(x_0, \dots, x_{d-1})$ is simply connected.

GAGA: Prove that for $d \geq 2$

$\text{Spec}(\overline{\mathbb{F}_q}[[x_0, \dots, x_{d-1}]]) \setminus V(x_0, \dots, x_{d-1})$ is simply connected.

Zariski-Nagata purity:

$f.\text{-ét} / \text{Spec}(\overline{\mathbb{F}_q}[[x_0, \dots, x_{d-1}]]) \simeq f.\text{-ét} / \text{Spec}(\overline{\mathbb{F}_q}[[x_0, \dots, x_{d-1}]]) \setminus V(x_0, \dots, x_{d-1})$
 simply connected (Hensel)

Rem: When E/\mathbb{Q}_p $B_{\overline{\mathbb{F}_q}}^{G=\text{trd}}$ is a diamond and the proof of the simple connectedness is much more involved.