# Structure of the Abel-Jacobi morphism and geometric local class field theory

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Lecture 4

$\boxed{\mathrm{Spa}(E)^\diamond}$ Sheaf of untilts over $E$

↳ on $\mathrm{Perf}_{\mathbb{F}_q}$

$S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathrm{Spa}(E)^\diamond(S) = \{(S^\#, \iota) \mid S^\# \in \mathrm{Perf}_E,\ \iota : S \xrightarrow{\sim} S^{\#,\flat}\}/\sim$

⟶ quickly said: "$S^\#$ is an untilt of $S$"

$\mathrm{Spa}(E)^\diamond$ is a pro-étale sheaf via Scholze's equivalence

$$(-)^\flat : \mathrm{Perf}_{S^\#} \xrightarrow{\sim} \mathrm{Perf}_{S}$$

of pro-étale sites. Moreover it induces

$$\boxed{\widetilde{\mathrm{Perf}}_E \xrightarrow{\sim} \widetilde{\mathrm{Perf}}_{\mathbb{F}_q}/\mathrm{Spa}(E)^\diamond}$$

En: $E = \mathbb{Q}_p$. $\widetilde{\mathrm{Perf}_{\mathbb{Q}_p}} = \widetilde{\mathrm{Perf}_{\mathbb{F}_q}} / \mathrm{Spa}(\mathbb{Q}_p)^\diamond$ char. 0 pro-étale topos
$\parallel$
localization of char. p pro étale topos.

# Construction of a morphism $\mathrm{Spa}(E)^\diamond \longrightarrow \mathrm{Div}^1$

## An adjunction: $E|\mathbb{Q}_p$

fully faithfull

$$\text{Perfect } \mathbb{F}_q\text{-alg.} \underset{(-)^\flat}{\overset{W_{\mathcal{O}_E}(-)}{\rightleftarrows}} \pi\text{-adic } \mathcal{O}_E\text{-algebras}$$

$$\begin{cases} \mathrm{Id} \xrightarrow{\sim} W_{\mathcal{O}_E}(-)^\flat \\ x \longmapsto ([x^{1/q^n}])_{n \geq 0} \end{cases}$$

Fontaine's $\theta$-morphism = adjunction morphism

$$\begin{cases} W_{\mathcal{O}_E}((-)^\flat) \xrightarrow{\theta} \mathrm{Id} \\ \sum\limits_{n \geq 0}' [x_n]\pi^n \longmapsto \sum\limits_{n \geq 0}' x_n^\# \pi^n \end{cases}$$

* $R$ perfectoid $\mathbb{F}_q$-algebra

$\xi = \sum\limits_{n \geq 0}' [x_n]\pi^n \in W_{\mathcal{O}_E}(R^\circ)$ is said to be primitive of degree 1 if $\begin{cases} x_0 \in R^{\circ\circ} \cap R^\times \\ x_1 \in (R^\circ)^\times \end{cases}$

Prop: (1) $R^\#$ untilt of $R$ over $E$, $R = R^{\#,b}$

$\theta : W_{O_E}(R^\circ) \twoheadrightarrow R^{\#,\circ}$ is surjective

with $\ker \theta = (\xi)$, $\xi$ primitive of degree 1.

(2) Reciprocally, if $\xi \in W_{O_E}(R^\circ)$ is primitive of degree 1 then $\boxed{W_{O_E}(R^\circ)/(\xi)\left[\frac{1}{\pi}\right]}$ is an untilt of $R$.

$$\Rightarrow \boxed{\mathrm{Spa}(E)^\diamond(R, R^+) = \{\text{deg. 1 primitive}\} / W_{O_E}(R^\circ)^\times}$$

* $\xi \in \mathcal{O}(Y_{R,R^+})$ and $\times\xi : \mathcal{O}_{Y_{R,R^+}} \hookrightarrow \mathcal{O}_{Y_{R,R^+}}$

$$\mathrm{Spa}(R^\#, R^{\#,+}) \xhookrightarrow{V(\xi)} Y_{R,R^+}$$

Cartier division

Rem:

$$\begin{tikzcd}
\mathrm{Spa}(R, R^+) \arrow[r, hook, "\pi = 0"] & \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \setminus V([\varpi]) \\
\mathrm{Spa}(R^\sharp, R^{\sharp,+}) \arrow[ur, hook, "\xi = 0"'] &
\end{tikzcd}$$

deformation of the Cartier divisor outside $\pi = 0$

$*$ $S \in \mathrm{Perf}_{\mathbb{F}_q}$ $\quad S^\sharp$ untilt of $S$ over $E$

$$\begin{tikzcd}
S^\sharp \arrow[r, hook] \arrow[dr] & Y_S \arrow[d] \\
& \mathrm{Spa}(E)
\end{tikzcd}$$

Cartier divisor locally on $S$ defined by $V(\xi)$

$\xi$ primitive of degree 1

$$\begin{tikzcd}
S^\sharp \arrow[r, hook] \arrow[dr, hook] & Y_S \arrow[d, two heads] \\
& X_S
\end{tikzcd}$$

Cartier divisor

easy

This defines a morphism

$$\begin{tikzcd}
\mathrm{Spa}(E)^{\diamond} \arrow[r] \arrow[dr] & \mathrm{Div}^1 \\
& \underbrace{\mathrm{Spa}(E)^{\diamond}/\varphi^{\mathbb{Z}}}_{\text{Frobenius of } \mathrm{Spa}(E)^{\diamond}} \arrow[u, dashed]
\end{tikzcd}$$

$\longleftarrow$ factorization

$$\boxed{\underline{\text{Th}}:\ \mathrm{Spa}(E)^{\diamond}/\varphi^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1}$$

$\underline{\mathrm{Div}^d}$ :

$$\boxed{\begin{array}{l}\underline{\text{Th}}:\ \Sigma^d : (\mathrm{Div}^1)^d \longrightarrow \mathrm{Div}^d \text{ induces} \\ \text{an isomorphism} \quad (\mathrm{Div}^1)^d/\mathfrak{S}_d \xrightarrow{\sim} \mathrm{Div}^d\end{array}}$$

quotient of pro-étale sheaves

$\longrightarrow$ proof technical: uses the notion of quasi-pro-étale morphism by Scholze. At the end this is reduced to prove that if $F/\mathbb{F}_q$ alg. closed:

$$\mathrm{Div}^1(F)^d/\mathfrak{S}_d \xrightarrow{\sim} \mathrm{Div}^d(F)$$

deduced from the following factorization theorem wrt. Fontaine:

$$\boxed{P = \bigoplus_{d\geq 0} \mathcal{O}(Y_F)^{\varphi=\pi^d} \text{ is graded factorial}}$$

with deg. 1 irreducible elements

$\rightsquigarrow \forall x \in P_d \setminus \{0\} \quad x = t_1 \dots t_d, \quad t_i \in P_1$

$(t_1, \dots, t_d)$ well defined up to permutation and multiplication by an element of $E^\times$.

$\Rightarrow \forall d \geq 1, \mathrm{Div}^d =$ diamond (alg. space for pro-étale top.)

$\overline{\mathbb{Q}}_\ell$ - local systems on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$:

$\breve{E} = \widehat{E^{un}}$

$\circlearrowleft \sigma$

$\sigma = \mathrm{Frob}$

$$\mathrm{Spa}(E)^\diamond \times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)} \mathrm{Spa}(\overline{\mathbb{F}}_q) = \mathrm{Spa}(\breve{E})^\diamond$$

$\circlearrowleft$

$\underbrace{\varphi_{E^\diamond} \times \mathrm{Id} \text{ and } \mathrm{Id} \times \varphi_{\overline{\mathbb{F}}_q}}_{\text{Composite of both partial Frob is absolute}} \longleftrightarrow \sigma$

Frob ↦ acts trivially on the étale topos.

$$\Rightarrow \overline{\mathbb{Q}}_\ell\text{-loc. sys.}/\mathrm{Div}^1_{\overline{\mathbb{F}}_q} = (\varphi_{E^\diamond} \times \mathrm{Id})\text{-eq. loc. sys. on } \mathrm{Spa}(E)^\diamond \times \mathrm{Spa}(\overline{\mathbb{F}}_q)$$

$$= G\text{-eq. loc. sys. on } \mathrm{Spa}(\breve{E})^\diamond$$

$$= \boxed{\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_E)}$$

## The Abel-Jacobi morphism

$d \geq 1$ $\qquad AJ^d : \mathrm{Div}^d \longrightarrow \mathrm{Pic}^d$

### Another formula for $\mathrm{Div}^d$:

$\mathbb{B}$ = pro-étale sheaf of ring on $\mathrm{Perf}_{\mathbb{F}_q}$

$\mathbb{B}(S) = \mathcal{O}(Y_S)$

sheaf of Fontaine's type of ring

$$\mathbb{B}^{\varphi=\pi^d} = \text{sheaf of relative global sections of } \mathcal{O}(d)$$

$$\mathbb{B}(S)^{\varphi=\pi^d} = H^0\big(X_S, \mathcal{O}(d)\big)$$

Pro-étale locally any $\mathcal{L}/X_S$ of deg. $d$ is isomorphic to $\mathcal{O}(d)$

$$\Rightarrow \quad \mathrm{Div}^d = \big(\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}\big) / \underbrace{\underline{E}^\times}_{\underline{\mathrm{Aut}}(\mathcal{O}(d))}$$

Thus

$$\boxed{AJ^d : \underbrace{\mathbb{B}^{\varphi=\pi^d} \setminus \{0\} / \underline{E}^\times}_{\mathrm{Div}^d} \longrightarrow \underbrace{\left[\mathrm{Spa}(\mathbb{F}_q) / \underline{E}^\times\right]}_{\mathrm{Pic}^d}}$$

One deduces

$$\boxed{\text{Th} : \forall d \geq 1 \quad AJ^d \text{ is a pro-étale locally trivial fibration in } \underbrace{\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}}_{\text{diamond}}}$$

# Geometric Langlands

Let us admit (difficult)

Th: $\forall d \geq 2$ (resp. $d \geq 3$ if $E|\mathbb{Q}_p$) $\mathbb{B}_{\overline{\mathbb{F}}_q} \setminus \{0\}$ is simply connected (any finite étale cover has a section).

Let $\chi : W_E \longrightarrow \overline{\mathbb{Q}}_\ell^\times$

$\rightsquigarrow \mathcal{E} =$ rk. 1 $\overline{\mathbb{Q}}_\ell$ - local system on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$

$\rightsquigarrow \mathcal{E}^{(d)} = \left(\Sigma^d_* \, \mathcal{E}^{\boxtimes d}\right)^{\mathfrak{S}_d}$ rk. 1 $\overline{\mathbb{Q}}_\ell$ - loc. sys on $\mathrm{Div}^d_{\overline{\mathbb{F}}_q}$

$\rightsquigarrow \mathcal{F}_\chi^{(d)}$ on $\mathrm{Pic}^d$ for $d \gg 0$, $\mathcal{E}^{(d)} = (AJ^d)^* \mathcal{F}_\chi^{(d)}$

$\rightsquigarrow$ group law on $\mathrm{Pic} \Rightarrow \mathcal{F}_\chi^{(d)}$ extends to all $d$'s to a local system on $\mathrm{Pic}$.

In particular $\boxed{\mathcal{E} \text{ descends along } AJ^1}$

What does it mean?

$$AJ^1 : \mathrm{Div}^1_{\overline{\mathbb{F}}_q} \longrightarrow \left[\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{E}^\times\right]$$

$$\text{"}\pi_1(AJ^1) : W_E \xrightarrow{f} E^\times\text{"}$$

no good theory of $\pi_1$ a priori but makes sense dually at the level of pullback of $\overline{\mathbb{Q}}_\ell$-local systems.

$\chi_{LT} : W_E \longrightarrow \mathcal{O}_E^\times$ Lubin-Tate character associated to $\pi$

$$f(\tau) = \chi_{LT}(\tau)^{-1} \pi^{v(\tau)}$$

$\tau \bmod \pi = \mathrm{Frob}_q^{v(\tau)}$

Thus $\forall \chi : W_E \to \overline{\mathbb{Q}}_\ell^\times$, $\chi$ factorizes through

$$\Rightarrow \boxed{f : W_E^{ab} \xrightarrow{\sim} E^\times}$$ ($f$ is easily surjective)

i.e. local class field theory.

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Proof that $B_{\overline{\mathbb{F}_q}}^{\varphi = \pi^d} \setminus \{0\}$ is simply connected when $d > 1$ for $\overline{\mathbb{F}_q}((t))$

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$(R, R^+)$ $\mathbb{F}_q$-alg. aff. perfectoide

$\|\cdot\| : R \to \mathbb{R}_+$ power mult. defining its topology

$$\mathcal{O}(Y_{R,R^+}) = \left\{ \sum_{n \in \mathbb{Z}} x_n \pi^n \,\middle/\, x_n \in R \ \forall \rho \in ]0,1[ \ \lim_{|n| \to +\infty} \|x_n\| \rho^n = 0 \right\}$$

$$(R^{\circ\circ})^d \xrightarrow{\sim} \mathcal{O}(Y_{R,R^+})^{\varphi = \pi^d}$$

$$(x_0, \dots, x_{d-1}) \longmapsto \sum_{i=0}^{d-1} \sum_{b \in \mathbb{Z}} x_i^{q^{-b}} \pi^{bd+i}$$

$$\Rightarrow B_{\overline{\mathbb{F}_q}}^{\varphi = \pi^d} \simeq \underbrace{\mathrm{Spa}\left(\overline{\mathbb{F}_q}[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right)}_{\text{perfect adic space not perfectoid since not analytic}}$$

$$\mathbb{B}^{\varphi=\pi^d}_{\overline{\mathbb{F}}_q} \setminus \{0\} = \mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right) \setminus V(x_0, \dots, x_{d-1})$$

= perfectoid quasicompact space

$$= \bigcup_{i=0}^{d-1} D(x_i)$$

$$\downarrow \text{ open perfectoid ball}$$

$$\mathrm{Spa}\left(\overline{\mathbb{F}}_q((x_i^{1/p^\infty}))\right)$$

$\rightsquigarrow$ Union of $d$ open perfectoid balls / different perfectoid fields

GAGA: Th: $A$ noetherian $I$-adic

$$\text{Finite étale}/\mathrm{Spec}(A) \setminus V(I) \xrightarrow{\sim} \text{Finite étale}/\mathrm{Spa}(A, A) \setminus V(I)$$

$$2\text{-}\varinjlim_n \text{f.ét}/\mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0^{1/p^n}, \dots, x_{d-1}^{1/p^n}]]\right) \setminus V(x_0, \dots, x_{d-1})$$

$$\downarrow \simeq \text{ Elkik decompletion}$$

$$\text{f.ét}/\mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right) \setminus V(x_0, \dots, x_{d-1})$$

$\rightsquigarrow$ Suffices to prove that $\mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0,\dots,x_{d-1}]]\right) \setminus V(x_0,\dots,x_{d-1})$ is simply connected.

GAGA: Prove that for $d \geq 2$

$$\mathrm{Spec}\left(\overline{\mathbb{F}}_q[[x_0,\dots,x_{d-1}]]\right) \setminus V(x_0,\dots,x_{d-1})$$

is simply connected.

Zariski-Nagata purity:

$$\underbrace{\mathrm{f.et}/\mathrm{Spec}\left(\overline{\mathbb{F}}_q[[x_0,\dots,x_{d-1}]]\right)}_{\text{simply connected (Hensel)}} \xrightarrow{\sim} \mathrm{f.et}/\mathrm{Spec}\left(\overline{\mathbb{F}}_q[[x_0,\dots,x_{d-1}]]\right) \setminus V(x_0,\dots,x_{d-1})$$

Rem: When $E|\mathbb{Q}_p$ $\quad \mathbb{B}^{\varphi=\pi^d}_{\overline{\mathbb{F}}_q} \setminus \{0\}$ is a diamond and the proof of the simple connectedness is much more involved.